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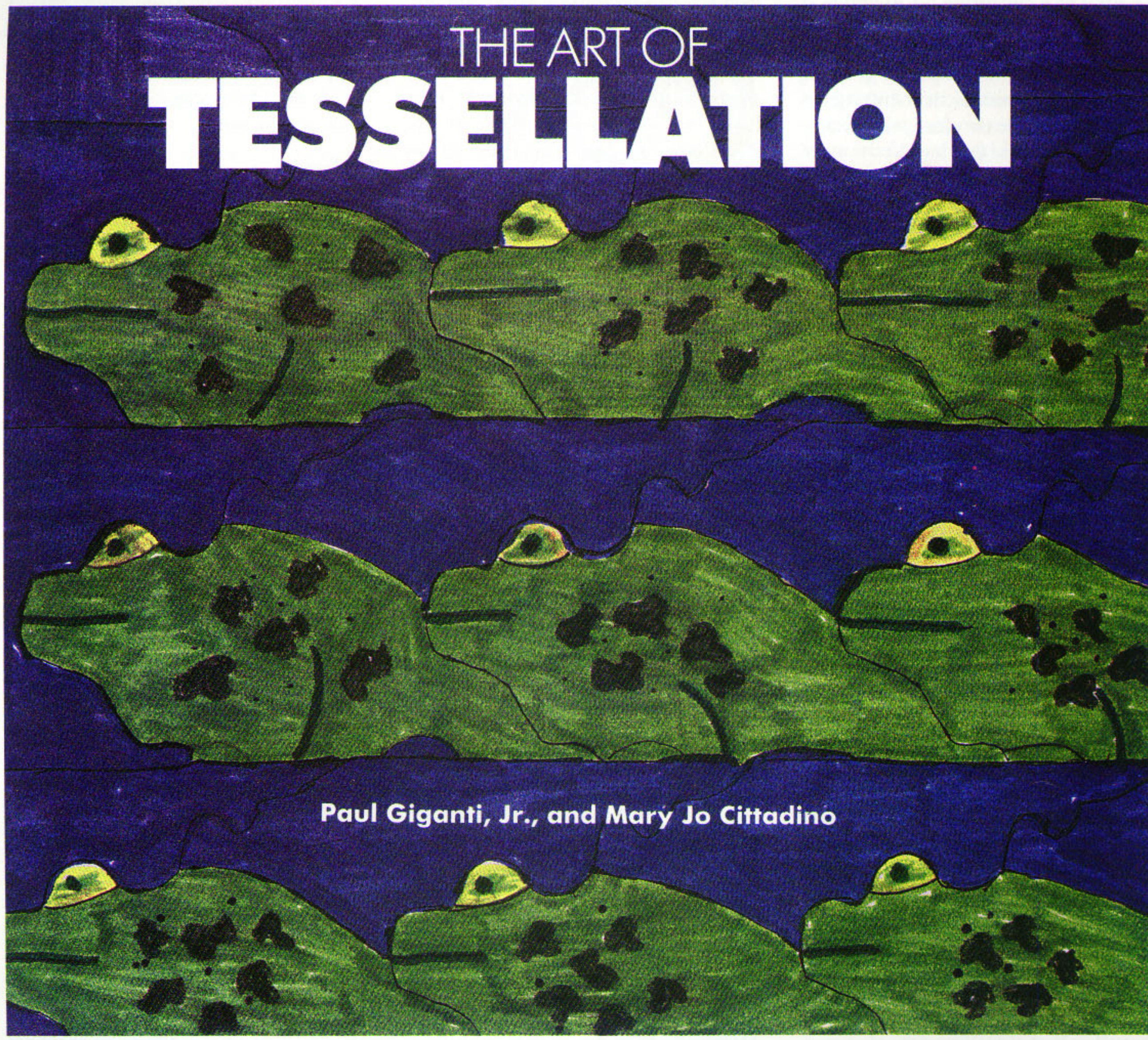
THE ART OF TESSELLATION

Paul Giganti, Jr., and Mary Jo Cittadino



Students need to experience the excitement and beauty of mathematics beyond numerical calculations. Introduce your students to tessellations, a project that combines mathematics and art. A tessellation is a tiling, made up of the repeated use of polygons and other curved figures to completely fill a plane without gaps or overlapping, just like the tiles on a kitchen or bathroom floor. See **figure 1**.

Students need to be encouraged to develop and explore their own artistic creativity and to exercise their spatial sense within mathematics. A project in student-designed tessellations can let students discover one aspect of mathematics that has fascinated creative persons throughout history. Leonardo da Vinci, M. C. Escher, and numerous others have used such mathematics in unique ways. Tessellation is a perfect example of how mathematics complements other disciplines.

Students in grade 3 through adult, with many different levels of ability in mathematics, can create tessellations. Tessellations do not depend heavily on numerical skills; all students have equal access and equal chances for success in creating beautiful artwork through tessellations. Many students become "experts" and "advisors" for the first time in the context of a

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mathematics class. This article relates our experience in presenting tessellations in both classrooms and teacher in-service workshops.

Preliminary Activities

Several preliminary activities will help your students understand the concept and process of covering a flat surface, or tessellating, before the final tessellation project begins.

- Use geometric figures (square, rhombus, triangle, hexagon, rectangle) to clarify the concept of tessellations. Give each student several congruent pieces, such as Pattern Block pieces. A few are included on the master-pattern sheet. Encourage them to see how they can arrange their figures to cover a surface completely. Let them make and share arrangements that do not cover a surface. Next, give each student just one polygon; have them place this polygon in the middle of a blank piece of paper and trace around it. Lining the polygon up with any side of the traced image, have them trace their polygon again. Students should repeat this process until they see that they can cover the entire paper in this manner. The instructor or students can also demonstrate their arrangement using Pattern Blocks on an overhead projector.

- Talk about examples of differently shaped tiles that students have seen covering bathroom and kitchen floors, bricks they have seen covering patios, and tilelike patterns they have seen on wrapping paper, fabric, wallpaper, and so on. (See **fig. 2**.) Look through magazines to find examples of pictures of tiling or tessellations. Have students make a collage of the pictures.

- Use tessellations as an opportunity to introduce or review geometric terms, such as *polygon, opposite, slide, rotate, hexagon,* and *congruent*. The mathematical vocabulary will help students understand the process of tessellation. (See the Glossary.)

- Students should experiment using geometric shapes of their own choosing, such as pentagons, other types of quadrilaterals, irregular triangles, and so on, to discover which ones tessellate and which ones don't. Make a "will work/won't work" wall chart on which students can paste up examples of figures that will tessellate and ones that will not. Remind them that uncovered "holes" or overlapping figures are not permissible.

- Find as many of the M. C. Escher art books as you can and share them with your class. Ask them to look for basic geometric figures, as well as pick out some recurring irregular features that Escher used in his tessellations. (See the Bibliography.)

Many of these preliminary activities take only minutes of class time and

FIGURE 1

A tessellation

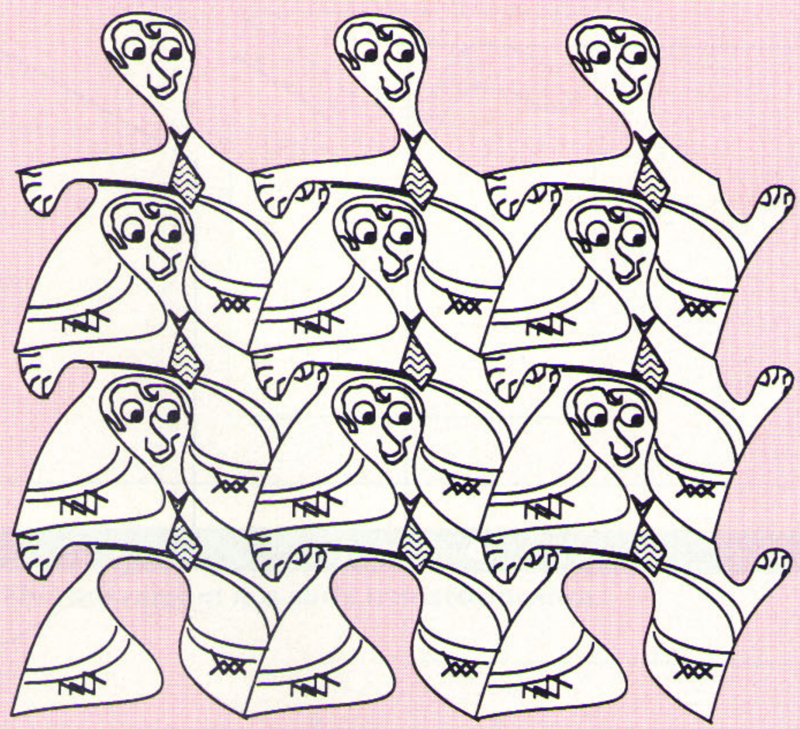
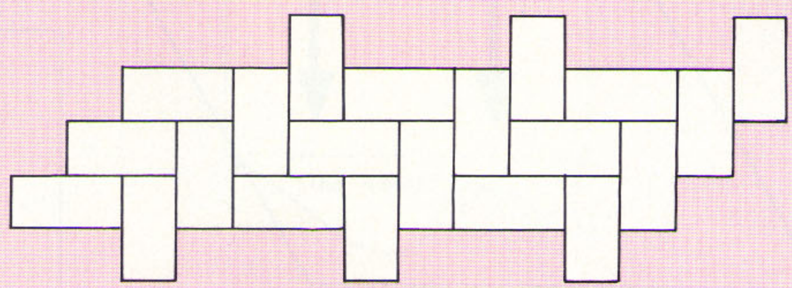
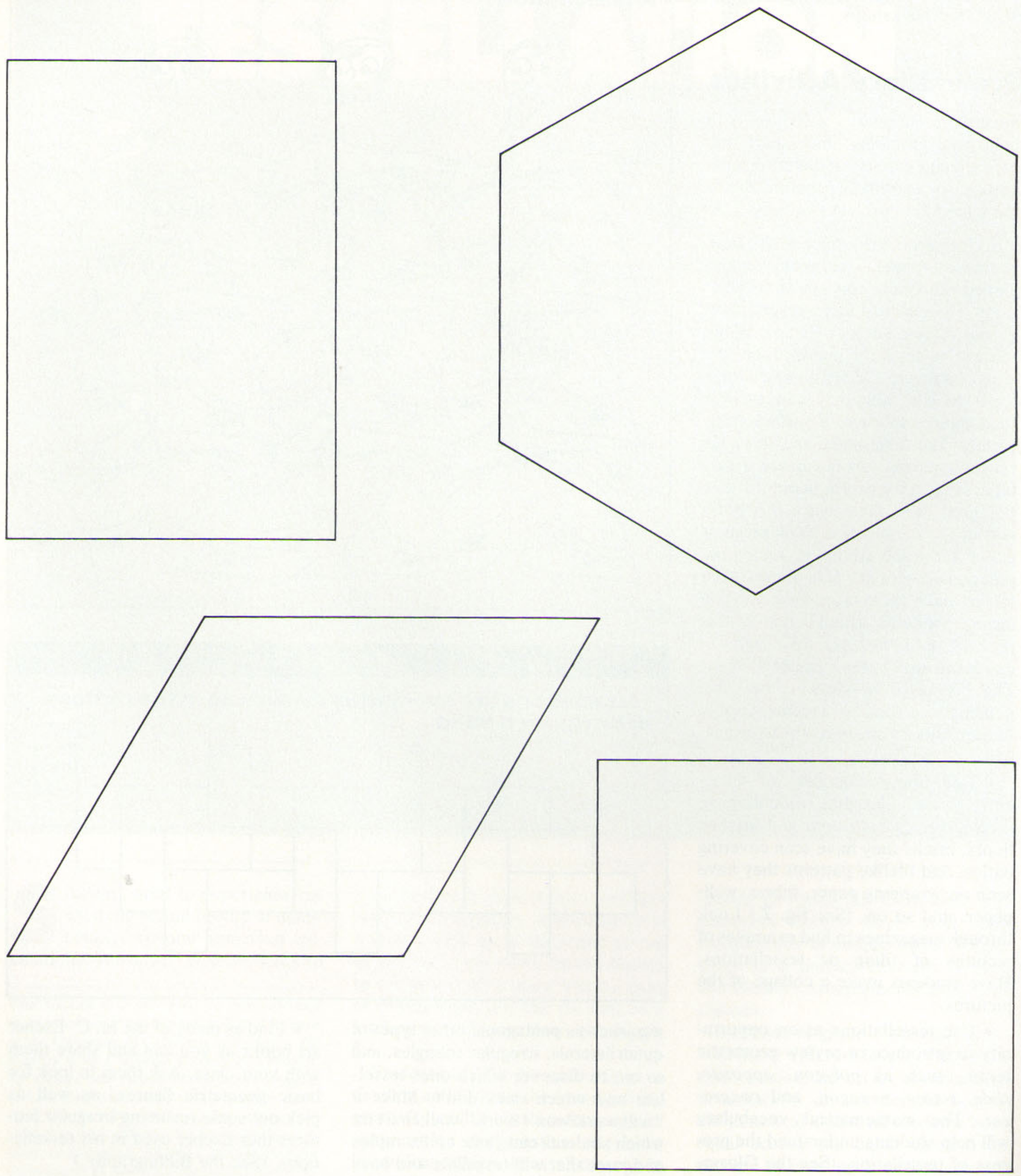


FIGURE 2

Tessellations are everywhere – such as in the pattern of bricks on a patio



MASTER-PATTERN SHEET



can be done concurrently with the study of other mathematics subjects, such as geometry and measurement. After completion of these activities, students will have a clear understanding of tessellations and be ready and eager to tessellate.

The "Nibble" Technique

Many geometric shapes tessellate. Polygons that tessellate can be altered to create irregularly shaped pictures that also tessellate. Using a simple "nibble" technique, students can easily create irregularly shaped tiles that will become tessellating pictures.

The "nibble" technique permits learners to change the shapes of figures. One overall rule to follow when altering a geometric shape to create a tessellation is that the shape must retain the same area as the original shape.

Translations or slides

This transformation is restricted to polygons—parallelograms and hexagons—whose opposite sides are parallel and congruent because an operation on one side always affects the opposite side.

- Using the master-patterns sheet provided, copy and cut from heavy card stock or old manila folders one of the shapes for each student. All students start with the square, which is the easiest shape to use, and experiment with the other shapes later.
- One side of this piece should first be colored completely with crayon to prevent students from inadvertently flipping the piece while moving or taping their "nibble."
- Demonstrate the "nibble" technique on the overhead projector. The technique that is easiest for beginners is to cut into one corner of a square, stressing that the choice is completely arbitrary and that you are starting and ending your cut at an adjacent corner. A "nibble" that is cut from adjacent corners will be easier to line up when matching sides. (See **fig. 3**.)
- Warn students that once they begin "nibbling" their square, no scraps can be discarded. Every piece has to be accounted for!
- Next demonstrate how to slide the newly cut "nibble" across to the congruent and parallel side. It must match the straight edges and corners before being attached to the side. Tape the "nibble" carefully and securely in its new home. (See **fig. 4**.)
- Since a square has four sides, a second "nibble" can be cut from one of the other pair of parallel sides and slid to the opposite side, once again matching the straight edges very carefully and then taping it into place. Re-

FIGURE 3

"Nibble" from one corner to an adjacent one.

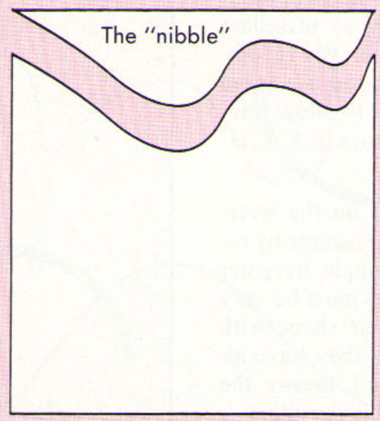
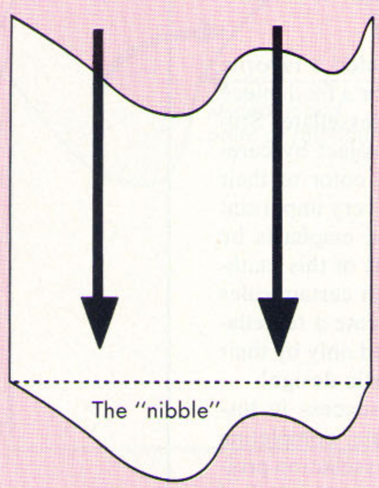


FIGURE 4

This movement is a slide transformation.



member: no trimming to fit! (See **fig. 5**.)

When students have finished "nibbling" and taping the sides of their square, they are ready to tessellate with the resulting shape. We recommend that at this point you use scrap paper just large enough to allow tracing the shape several times (8.5″ × 11″ or 9″ × 12″).

• Demonstrate again on the overhead projector how to tessellate by repeated tracing of a sample irregular tile. Stress that students must be very careful in lining up their shape with the sides of the shapes they have already traced. "Sliding" means the shape itself slides along the plane—up, down, left, or right. Care must also be taken not to flip the piece over or rotate it while tracing. Keeping the color side up at all times should help.

• Students can practice the nibble-and-slide technique for the other geometric shapes. The more sides, the more sides they can alter (e.g., the hexagon will allow them to modify three pairs of sides).

When they have created a favorite shape, they are ready for a fresh piece of paper on which to tessellate. Students complete their project by carefully adding detail and color to their tessellated design. It is very important that adequate time and emphasis be given to the artistic part of this mathematical lesson. Though certain rules must be followed to create a tessellation, students are limited only by their imaginations in its artistic design!

Once students find success in tessellating on a small scale, completing a larger, poster-sized tessellation project can give students a great deal of satisfaction.

FIGURE 5

Sides can be altered in this manner.

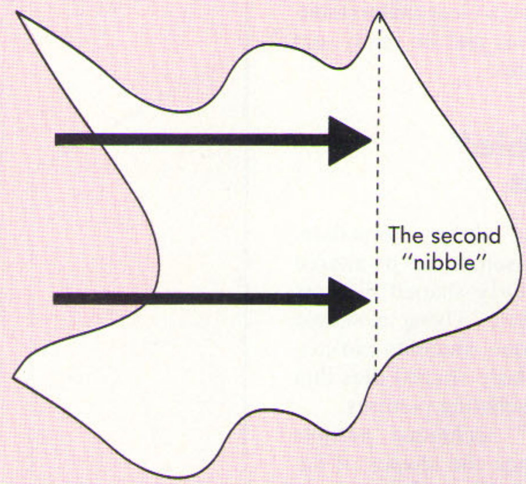
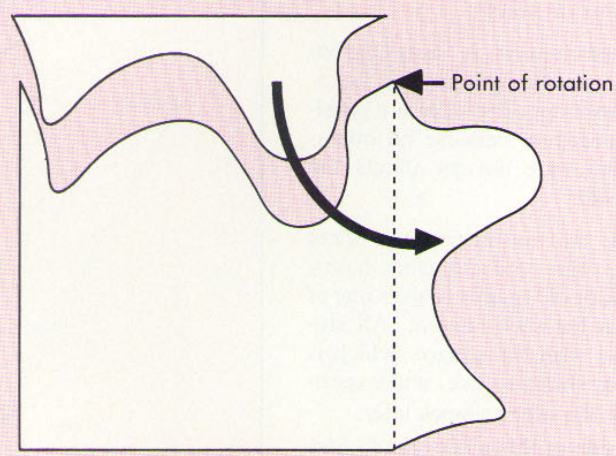


FIGURE 6

This transformation is a rotation.



Rotations or turns

This transformation is restricted to polygons—triangles, parallelograms, and hexagons—with adjacent sides that are congruent. Students should master the slide technique before trying rotations or turns.

• Begin again with the square. Give students another square cut from tagboard or a manila folder and have them color one side to distinguish a front and back. Using the "nibble" technique, students cut out a "nibble" from corner to corner of their square, but this time they rotate the "nibble" at its endpoint to an adjacent side of their square, not an opposite side. Again they tape the piece securely into place after carefully matching the straight edges. (See **fig. 6**.)

• The students alter another side of the square and rotate this "nibble" from its endpoint to the adjacent side of the square and tape. We strongly recommend that all the sides of the square be altered differently to avoid confusion when tessellating. (See **fig. 7**.)

• Students now rotate or turn the shape as they move and trace the tile to cover the plane.

• Students may need to experiment to tessellate shapes altered in this manner. For some shapes, flipping the pieces is necessary to tessellate the plane.

Rotation or turn at midpoints

Students use a ruler to mark the midpoint on each of the four sides of the square, then proceed to make a "nibble" from one corner to the marked midpoint of a side of a square. They rotate this piece about the side's midpoint onto the remaining half of this same side, then tape. They repeat this procedure for all four sides of the square. (See **fig. 8**.)

Experimenting with variations

- Students can experiment with beginning their cut from one corner and terminating it at any point before reaching the other corner.

- Another variation is to avoid corners at the beginning and end of one's cut on a side. This approach requires using a ruler to determine exactly where to affix the piece on the opposite or adjacent side. For an altered shape to tessellate the plane, it is imperative that it match exactly the position from where it came.

FIGURE 7

All sides of the original shape have been altered.

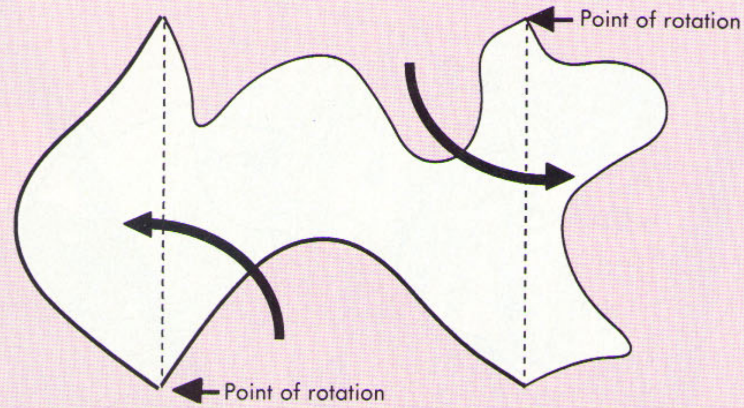
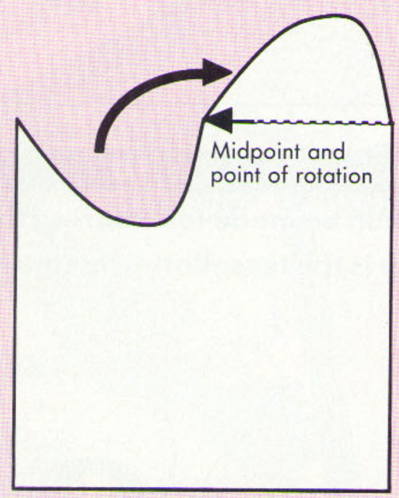


FIGURE 8

A rotation around the midpoint of a side



The Art in Tessellations

To create a beautiful and interesting tessellation, students must understand that tessellation is a true combination of art and mathematics; both are equally important. To facilitate students' successful artistic tessellations, the teacher should stress the artistic part of tessellating as much as the mathematical part. The following suggestions will help your students think like artists.

- Share examples of your own tessellations. Nothing encourages students more than seeing projects their teachers have done themselves. See the tessellation by the author in **figure 9**.

- Advise your students that thought, care, and time are important ingredients in any good tessellation; careful work or a rushed job show up quite clearly in tessellations.

- Introduce the artistic side of tessellations by using blank irregular tessellations, such as those in **figure 10,** on the overhead projector. Put a single transparent sample tile on the projector and brainstorm with the whole class about objects or animals into which the tile could be transformed by adding lines, marks, or color. Try rotating the tile to see different things.

- Using the tessellation worksheets, have students add just the artwork to make a finished tessellation. Make sure they share their results with the whole class. (See student worksheets A and B.)

- Explain to students that the irregular tessellation that they create may not look like anything recognizable at first. It is very difficult to create a specific object right from the beginning. Instead they must use their imagination and add the artwork to make their tile look like something. If students

FIGURE 9

Author's tessellation

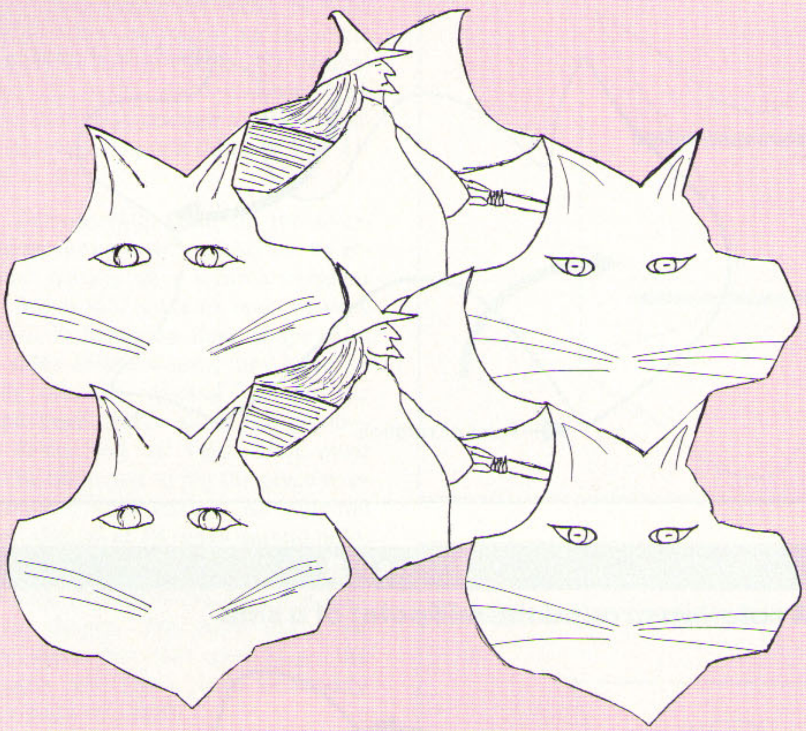


FIGURE 10

Many choices can be made for coloring this shape. One possibility is the tessellation in figure 1.

have difficulty seeing a design in their tiles, encourage them to get suggestions from classmates; a student will often see a wonderful possibility in someone else's shape.

- As they trace and draw their own tessellation projects, students should make very light lines so that changes will be easy to make. After all the tracing is done—students may need to "fudge" a little here and there—have them go back over all the lines freehand to make them darker and to correct any errors in tracing.
- Color makes all the difference in successful tessellations, as it does in most artwork. Using contrasting colors for adjacent tiles, like on a checkerboard, is one very effective technique.
- Some students may wish to experiment by coloring each tile or a row of tiles with a different design. (See student illustrations in **fig. 11**.)
- Keep reminding your students that they are indeed creating a work of art!

What to Look for When Shapes Don't Tessellate

- Often students will inadvertently flip over the "nibble" piece before reattaching. This mistake can be avoided if students color one side of their shape first.
- Sometimes students do not tape their "nibble" piece directly opposite, or parallel to, the spot from which it was cut. This constraint is important when doing slides.
- If a "nibble" involves two sides at a corner and is rotated to an adjacent side, the tile will not tessellate.
- If students use too much tape, they will extend the area of the original shape, and it will not tessellate.

Conclusion

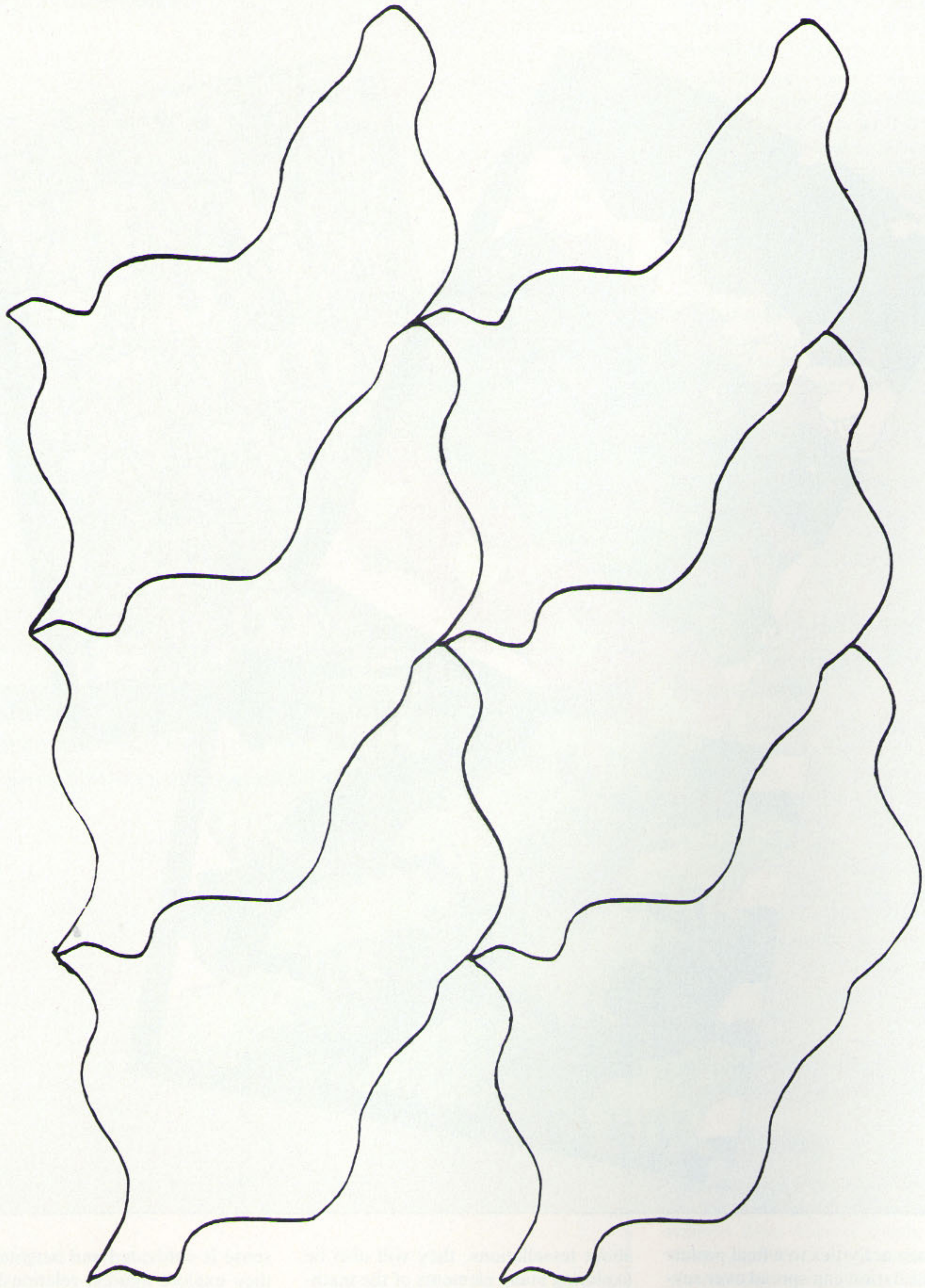
Students need to understand that not all mathematics activities take fifty minutes from start to finish. Tessellations represent an example of project-oriented mathematics. Though a complete tessellation project from

FIGURE 11

preliminary activities to a final poster-sized tessellation can spread over several weeks while students are learning about tessellations, they will also be exploring many elements of the mathematics curriculum. Their spatial sense is cultivated and heightened as they explore figures, relationships in figures, and effects of changing fig-

STUDENT WORKSHEET A

Name ____________________



STUDENT WORKSHEET B Name ____________________

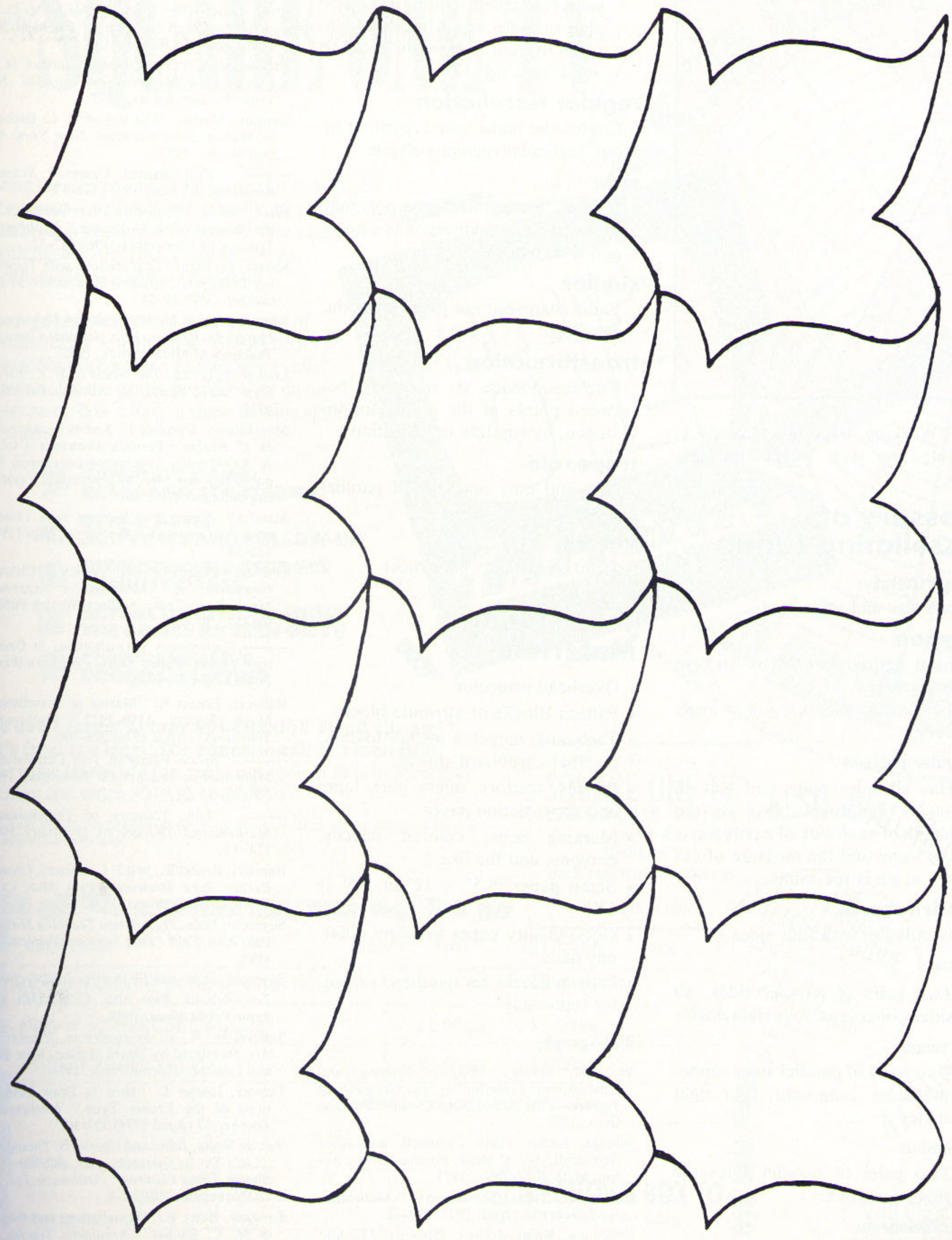
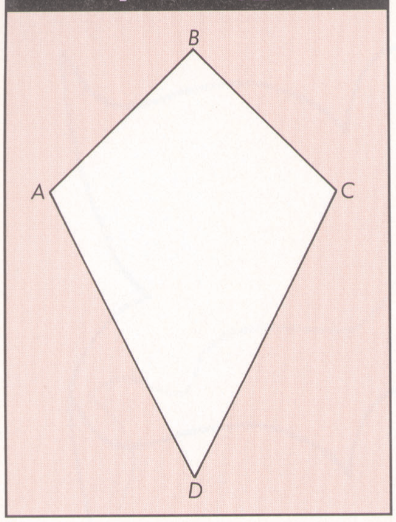


FIGURE 12



ures. Working with tessellations is valuable and well worth the time spent.

Glossary of Tessellating Terms

congruent

Same size and same shape

polygon

Closed figures formed by straight line segments

poly = many (Greek); *gon* = angle (Greek)

regular polygon

Has all sides congruent and all angles congruent; that is, the length of each side of a polygon is the same and the measure of every angle is the same

quadrilateral

Any polygon with four sides

square

Two pairs of parallel sides, all sides congruent, four right angles

rectangle

Two pairs of parallel sides, opposite sides congruent, four right angles

rhombus

Two pairs of parallel sides, all sides congruent

parallelogram

Two pairs of parallel sides

kite

Two distinct pairs of adjacent sides congruent (note that flipping or reflecting a triangle will result in a kite), $AB = BC$ and $AD = CD$ (see **fig. 12**)

regular tessellation

Covers the plane with repetition of one particular regular polygon

side

Straight line segment of a polygon, for example, a dodecagon is a polygon with twelve sides

similar

Same shape but not necessarily the same size

transformation

Correspondence or matching between points of the plane, for instance, by rotation or reflection

trapezoid

One and only one pair of parallel sides

vertex

Point where two sides meet

Tessellating Materials

- Overhead projector
- Pattern Blocks or attribute blocks
- Tagboard, recycled manila folders, or other cardboard sheets
- Pencils, scissors, rulers, glue, tape, and construction paper
- Marking pens, colored pencils, crayons, and the like
- Scrap paper (8.5″ × 11″ or 8.5″ × 14″)
- Good-quality paper in many different sizes
- Pattern Blocks for overhead projector (optional)

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